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EE 364 - Spring 2026

Week 3
Supplemental Notes

3 types of proof

1. Direct : $P \rightarrow Q$

   assume P, derive Q

2. Indirect : "reductio ad absurdum" (reduce to absurdity)

   assume $P \wedge \sim Q$, derive $C \wedge \sim C$

   $P \rightarrow Q \overset{M.I.}{=} \sim P \vee Q = \sim(P \wedge \sim Q)$

   $\therefore \sim(P \rightarrow Q) = \boxed{P \wedge \sim Q}$

Thrm: $(P \rightarrow (Q \,\&\, \sim Q)) \rightarrow \sim P.$

P	Q	P	$\rightarrow$	$(Q \wedge \sim Q)$	$\Longrightarrow$	$\sim P$
1	1	1	0	0	1	0
0	1	0	1	0	1	1
1	0	1	0	0	1	0
0	0	0	1	0	1	1

$\therefore$ Valid

Ex: Thrm: $\forall \varepsilon > 0 \quad a \le b + \varepsilon \longrightarrow a \le b.$

Prf: Suppose not. i.e. $\forall \varepsilon > 0 : (a \le b + \varepsilon) \wedge (a > b)$

$a = \frac{a}{2} + \frac{a}{2} \overset{\text{Assum.}}{>} \frac{a}{2} + \frac{b}{2} = b - \frac{b}{2} + \frac{a}{2} = b + \frac{a-b}{2}$ $\quad \varepsilon' > 0$ since:

$a > b \longleftrightarrow a - b > 0$

$= b + \varepsilon' \wedge \varepsilon' > 0$ $\quad \therefore \frac{a-b}{2} > 0$

$\therefore \varepsilon' > 0$

$\therefore \quad a > b + \varepsilon' \quad$ w/ $\varepsilon' > 0$

$\rightarrow \leftarrow$ contradiction: since $a \le b + \varepsilon \quad \forall \varepsilon > 0$

Thrm: $Q \wedge \sim Q \longrightarrow P$

(true AND false imply anything)

IMAC example: Finite sigma-algebra

Note: HW1/2

Sample space X contains four elements $X = \{a, b, c, d\}$. So its power set or set of subsets 2^X contains 2^4 elements. Thus the power set 2^X gives rise to 2^{16} subcollections of sets. Suppose that $A \subset 2^X$ is one of these 2^{16} set subcollections of 2^X and that A contains the following 7 sets:

$$A = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

Can we define a probability space (X, A, P) using the set collection A if P is a probability measure?

(use IMAC format to answer)

Soln:

$\underline{\text{I}}$: sigma-algebra (s.a.)

$\underline{\text{M}}$: $\mathcal{Q} \subset 2^X$ is a sigma-algebra iff $\mathcal{Q}$ is $\underline{\text{CUT}}$:

$$\begin{cases} C: A \in \mathcal{Q} \Longrightarrow A^c \in \mathcal{Q} \\ U: A_1 \in \mathcal{Q}, A_2 \in \mathcal{Q}, \ldots \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{Q} \\ T: X \in \mathcal{Q} \end{cases}$$

$\underline{\text{A}}$:

$\underset{\sim}{C}$: $\{a\}^c = \{b,c,d\} \in \mathcal{Q}$

$\{b\}^c = \{a,c,d\} \in \mathcal{Q}$.

$\{a,b\}^c = \{c,d\} \notin \mathcal{Q}$

$\left(\begin{array}{l} \{b,c,d\}^c = \{a\} \in \mathcal{Q} \\ \{a,c,d\}^c = \{b\} \in \mathcal{Q} \\ \phi^c = X \in \mathcal{Q}, \quad X^c = \phi \in \mathcal{Q} \end{array} \right)$

$\therefore$ $\mathcal{Q}$ is not closed under complement

because $\{a,b\}^c \notin \mathcal{Q}$

U: $\{a\} \cup \{b\} = \{a,b\} \in \mathcal{Q}$

$\{a\} \cup \{a,b\} = \{a,b\} \in \mathcal{Q}$

$\{a\} \cup \{a,c,d\} = \{a,c,d\} \in \mathcal{Q}$

$\{a\} \cup \{b,c,d\} = X \in \mathcal{Q}$

$\{b\} \cup \{a,b\} = \{a,b\} \in \mathcal{Q}$

$\{b\} \cup \{a,c,d\} = X \in \mathcal{Q}$

$\{b\} \cup \{b,c,d\} = \{b,c,d\} \in \mathcal{Q}$

$\emptyset \cup S = S \in \mathcal{Q}$ if $S \in \mathcal{Q}$

$X \cup S = X \in \mathcal{Q}$ if $S \in \mathcal{Q}$

$\therefore$ $\mathcal{Q}$ is closed under union.

T: $X \in \mathcal{Q}$.

$\therefore$ $\mathcal{Q}$ is $\cancel{C}$UT

$\therefore$ $\mathcal{Q}$ is not a sigma-algebra.

C: **NO.**

(X, A, P) cannot be a probability space

because A is not a sigma-algebra.

Discrete Sample Spaces

Finite or denumerable outcomes

$\hookrightarrow$ can put into 1-to-1 correspondance w/ $\mathbb{N}$

Ex: $\Omega = \{a_1, a_2, \dots, a_n\}$ finite sample space

Consider "equally likely outcomes"

$$P[\{a_i\}] = \frac{1}{n} \quad \forall i = 1, \dots, n$$

If event consists of k distinct outcomes:

$$P[B] = P[\{a_1', a_2', \dots, a_k'\}] = P[\{a_1'\}] + \dots + P[\{a_k'\}]$$

$$= \frac{k}{n}$$

Ex: 3 coin flips : 8 outcomes

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Assume equiprobable : $P[xxx] = \frac{1}{8}$

$P[2 \text{ heads}] = P[HHT \text{ or } HTH \text{ or } THH] = \frac{3}{8}$

Counting methods and Combinatorics

In finite sample space w/ equiprobable outcomes.

$$P[\text{event}] = \frac{\text{distinct \# of outcomes that match event}}{\text{total \# distinct outcomes}}$$

Ex: multiple choice exam, 10 questions, 4 options each

$$\# \text{ outcomes} = \underbrace{4 \times 4 \times \dots \times 4}_{10} = 2^{20} \approx 1 \text{ million}$$

Computer Problem: Birthday Paradox

Q: How large of group of k people to have prob. at least 0.5 of having at least two people with same birthday

(assume: 365 days/year, independent birthdays, equally likely per day)

Let A_k = event at least 2 people in group w/ k share a birthday.

define $q_k = P[\underbrace{A_k^c}_{\text{no common birthday in group of k.}}]$

$q(1) = 1$

$q(2) = q(1) \cdot \frac{364}{365} = \frac{\cancel{365}}{365} \cdot \frac{364}{\cancel{365}} = \frac{364}{365}$

$q(3) = q(2) \cdot \frac{363}{365} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$

$\vdots$

"365 fall k"

$q(k) = q(k-1) \cdot \frac{365-k+1}{365} = \frac{365 \cdot 364 \cdot (365-k+1)}{365^k} = \frac{(365)_k}{365^k}$

Computer: $q(22) = 0.524$

$q(23) = 0.493$ $\therefore k \geq 23$ people.

Random sampling (ordered)

- with replacement
  - choose k from n distinct elements.
  - select 1 at a time and "replace" after each selection

    ∴ duplicate samples possible

  \# outcomes: ordered k-tuple w/ n-options for each position

  $$= \underbrace{n \cdot n \cdots n}_{k} = n^k$$

Ex: multiple choice exam w/ 4-options per question.

- without replacement
  - choose k from n distinct elements
  - select 1 at a time & "remove" after each selecting

    ∴ no duplicate samples possible

$n_1 = n$

$n_2 = n-1$

$n_3 = n-2$

$\vdots$

$n_k = n-k+1$

∴ \# outcomes = $n \cdot (n-1)(n-2) \cdots (n-k+1)$

(note $k \leq n$)

Ex: Drawing balls \#1-60 from urn, don't replace

Special case: permutation.

↳ Sampling w/o replacement, $k = n$

\# outcomes $= n \cdot (n-1)(n-2) \cdots (2)(1)$

$$= \boxed{n!}$$

Defn: Permutation = Self-Bijection

- $f: S \to S$
- f: 1-to-1 (injection)
- f: onto (surjection)

Fundamental Rule of counting

arrangement = $\boxed{n_1 \mid n_2 \mid \cdots \mid n_k}$

$$\therefore \left| \text{arrangements} \right| = \prod_{k=1}^{n} n_k$$

$$\left(= n^k \text{ if } n_k = n \ \forall k \right)$$

$P(n,k)$ = # permutations of length k on n elements

$$n \cdot (n-1) \cdots (n-(k+1)) \cdot \frac{(n-k)!}{(n-k)!}$$

$$= \frac{n!}{(n-k)!} \quad \text{for } k \leq n.$$

Defn: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ for $n \in \{0, 1, 2, \ldots\}$.

$(0! = 1)$

Fact: (Stirling's Approximation)

$$\ln n! \approx n \cdot \ln n - n$$

$$\approxeq n \cdot \ln n$$

Ex: $n! < n^n$, for $n > 2$

$\therefore \ln n! < \ln n^n = n \cdot \ln n$

n^{th} term

$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} > \frac{n^n}{n!}$ ⟵ since $\sum_{k=0}^{\infty} \frac{n^k}{n!} = \frac{n^0}{0!} + \frac{n}{1!} + \frac{n^2}{2!} + \dots + \frac{n^n}{n!} + \dots$

$\therefore n! > \frac{n^n}{e^n}$

$\therefore \ln n! > n \cdot \ln n - n$

$\therefore n \cdot \ln n - n < \ln n! < n \cdot \ln n$

$\therefore 1 - \frac{1}{\ln n} < \frac{\ln n!}{n \cdot \ln n} < 1$

$\therefore \lim_{n \to \infty} \frac{\ln n!}{n \cdot \ln n} = 1$ $\qquad \therefore \ln n! \sim n \cdot \ln n$

Note: $\ln n! = \ln \prod_{k=1}^{n} k = \sum_{k=1}^{n} \ln k \sim \int_{x=1}^{n} \ln x \, dx$

pick $u = \ln x$ $\qquad dv = dx$

$du = \frac{1}{x}$ $\qquad v = x$

$$= x \cdot \ln x \Big|_{x=1}^{x=n} - x \Big|_{x=1}^{x=n} = n \cdot \ln n - n + 1 \sim n \cdot \ln n - n$$

Unordered Sampling

order doesn't matter, i.e., $(a, b) \sim (b, a)$

- without replacement
  - choose k from n w/o replacement
  - outcome order does <u>not</u> matter

<u>Ex</u>: "Standard lottery"

Draw 6 balls — "lottery numbers"

any order wins

$\therefore$ 60 balls choose 6 : $60 \cdot 59 \cdots 55 = \dfrac{60!}{54!}$ — $n!$, $(n-k)!$

but each outcome has : $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ equivalent outcomes

$\therefore$ # outcomes $= \dfrac{60!}{54!\,6!}$

<u>Defn</u>: $\dfrac{n!}{(n-k)!\,k!} = \binom{n}{k}$ is the <u>binomial coefficient</u>

- with replacement
  - choose k from n w/ replacement.
  - outcome order does <u>not</u> matter.

<u>Ex</u>: "Pizza toppings"

$k \to$ 4 toppings
+
$n \to$ 5 choices (duplicate OK)

A	B	C	D	
XX		X	XX	← outcome #1
X	XX	X	X	← outcome #2

Shorthand: XX | | X | XX #1 — 5 "X" + 3 "|"

X | XX | X | X #2 — "stars and bars"

$\therefore$ every outcome unordered w/ replacement is equivalent to a unique ordered sample of "X" and "|"

 k $n-1$

$\therefore$ # distinct outcomes : sample w/o replacement from urn w/ k-black balls and $(n-1)$ white balls.

($\therefore$ total $= n-1+k$ balls)

Ex : (cont) How many distinct ordered permutations of k black balls and $n-1$ white balls?

Consider balls # $1, 2, \ldots, n-1+k$ (no color)

then choose k, put "X" there. "|" else.

$$\frac{(n-1+k)!}{k!\,(n-1)!} \qquad \binom{n-1+k}{k} = \binom{n-1+k}{n-1} \qquad \frac{(n-1+k)!}{(n-1)!\,k!}$$

choose k for "X" pos. $\qquad$ choose $n-1$ for "|" pos.

Counting method summary

Ordered \ Replacement	with	without
yes	n^k Ex: # ways to answer multiple choice test	$n(n-1)\cdots(n-(k-1))$ Ex: How many ways to rank k students in class with n students
no	$\dfrac{(n-1+k)!}{(n-1)!\,k!}$ Ex: # different pizzas w/ k-toppings (duplicate OK)	$\dfrac{n!}{k!\,(n-k)!}$ Ex: # ways to draw k "lottery numbers"

$$\binom{n-1+k}{n-1} = \binom{n-1+k}{k} \qquad \binom{n}{k}$$

Binomial Coefficients

$P(n,k)$ = # permutations of length k on n elements

$$= \frac{n!}{(n-k)!} \quad \text{for } k \le n.$$

$C(n,k)$ = # combinations of length k on n elements

$$= \binom{n}{k} = \frac{n!}{(n-k)!\,k!} \quad \text{for } k \le n.$$

Note: $\binom{n}{0} = 1 \longleftrightarrow \emptyset$

$\binom{n}{n} = 1 \longleftrightarrow \Omega$

Pascal's triangle

$$\binom{3}{1} \qquad \binom{3}{2} \qquad \binom{4}{2}$$

$$\begin{array}{c|ccccccccccc}
0 & & & & & & 1 & & & & & \\
1 & & & & & 1 & & 1 & & & & \\
2 & & & & 1 & & 2 & & 1 & & & \\
3 & & & 1 & & 3 & & 3 & & 1 & & \\
4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & &
\end{array}$$

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = \dots$

$(x+y)^5 = \dots$

note: # ways of choosing _ _ _ w/ 2 "x" and 1 "y"

just a hypothesis now
proof is binomial theorem

for $\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$

Thrm: (Pascal's Formula) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Prf: $\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} + \frac{(n-1)!}{k!\,(n-1-k)!}$

$$= \frac{k}{k} \cdot \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!} \cdot \frac{n-k}{n-k}$$

$$= \frac{(n-1)!\,k}{k!\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,(n\cancel{-k+k})}{k!\,(n-k)!} = \frac{(n-1)!\,n}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k}$$

QED.

Review: how to change the INDEX of summation

Show: $\sum_{k=1}^{n} a_{k-1} = \sum_{j=0}^{n-1} a_j$

Technique: $\sum_{k=1}^{n} a_{k-1} = \sum_{k=1}^{k=n} a_{k-1}$ put $j = k-1$ $\therefore k = j+1$

$$= \sum_{j+1=1}^{j+1=n} a_j$$

$$= \sum_{j=0}^{j=n-1} a_j$$

$$= \sum_{j=0}^{n-1} a_j$$

QED.

☆☆☆

Thrm: (Binomial Theorem)

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$$

Prf: (by induction on $n = 1, 2, 3, \ldots$)

Basis: $n = 1$ $\quad\therefore (p+q)^1 = p+q.$

$$\sum_{k=0}^{1} \binom{1}{k} p^k q^{n-k} = \binom{1}{0} p^0 q^1 + \binom{1}{1} p^1 q^0$$
$$= p+q.$$

QED (basis)

Induction Step $\quad$ Induction hypothesis: $(p+q)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$

$$\therefore (p+q)^n = (p+q)(p+q)^{n-1}$$
$$\overset{IH}{=} (p+q) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$$
$$= \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k+1} q^{n-k-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-k}$$
$$= \underbrace{\sum_{j=1}^{n} \binom{n-1}{j-1} p^j q^{n-j}}_{j=k+1 \quad \therefore k=j-1} + \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{n-j}}_{j=k}$$
$$= \left[\underbrace{\binom{n-1}{n-1}}_{=1} \underbrace{p^n q^{n-n}}_{p^n} + \sum_{j=1}^{n-1} \binom{n-1}{j-1} p^j q^{n-j} \right] +$$
$$\left[\sum_{j=1}^{n-1} \binom{n-1}{j} p^j q^{n-j} + \underbrace{\binom{n-1}{0}}_{=1} \underbrace{p^0 q^{n-0}}_{q^n} \right]$$
$$= p^n + \sum_{j=1}^{n-1} \binom{n-1}{j-1} p^j q^{n-j} + \sum_{j=1}^{n-1} \binom{n-1}{j} p^j q^{n-j} + q^n.$$
$$= p^n + \sum_{j=1}^{n-1} \left[\underbrace{\binom{n-1}{j-1} + \binom{n-1}{j}}_{=\binom{n}{j}} \right] p^j q^{n-j} + q^n$$

$$= p^n + \sum_{j=1}^{n-1} \binom{n}{j} p^j q^{n-j} + q^n$$

$$= \binom{n}{n} p^n q^{n-n} + \sum_{j=1}^{n-1} \binom{n}{j} p^j q^{n-j} + \binom{n}{0} p^0 q^{n-0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

QED.

3 Corollaries

1. $p = q = 1$ $\qquad \therefore \sum_{k=0}^{n} \binom{n}{k} = 2^n$

since $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

2. $p + q = 1$ w/ $p \geq 0,\ q \geq 0$ $\qquad (\therefore q = 1-p)$

$$\therefore \sum_{k=0}^{n} P(X=k) \overset{\text{binomial}}{=} \sum_{k=0}^{n} b(n,k,p) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$\overset{BT}{=} (p+q)^n = 1^n = 1 \qquad \left(\text{w/ } b(n,k,p) = \binom{n}{k} p^k q^{n-k}\right)$$

$\therefore$ Binomial $b(n,k,p)$ is a probability density

3. $p = -1$ and $q = +1$ $\qquad \therefore \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$